Senior Honors Thesis

The Prime Number Theorem and Its Connection with the Riemann Hypothesis

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Abstract: The topic of the paper is the Prime Number Theorem and its connection with the

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I. The Prime Number Theorem

1. Introduction

Due to their random behavior and their great importance in number theory and abstract algebra, prime numbers have been an interesting topic for mathematicians to study since the

Euclid's "Elements". Then, in 1737, Euler proved the divergence of the harmonic series of primes. Towards the end of 18^{th} century, two mathematicians, Gauss and Legendre, working

Below is a table taken from Edwards of some values of $\pi(x)$ and its approximations:

x	$\pi(x)$	$x/\ln x$	$Li(x)$

2. Introduction of the Riemann Zeta function

a. The Zeta function:

[redacted]

and n ranges over all positive integers, Riemann considered s as a complex variable and studied the function on the new complex plane.

Using the factorial function and contour integration, Riemann derived a formula for $\sum_{n=1}^{\infty} n^{-s}$ that "remains valid for all s"[2]

[redacted]

b. Characters of the Zeta function:

In his paper, after defining the zeta function, Riemann analyzed its properties. In this section, a few important properties and sketches of their proofs are shown.

[illegible]

$s = 1$.

Proof:

Proof:

By making use of the basic properties of the factorial function, the formula (4) can be rewritten as

$$\Pi\left(\frac{s}{2}-1\right)\pi^{-s/2}\zeta(s)=\Pi\left(\frac{1-s}{2}-1\right)\pi^{-(1-s)/2}\zeta(1-s). \qquad (5)$$

Since the value of the function on the left-hand side remains unchanged when s is

[redacted]

Q. E. D.

Note: In his paper, Riemann also showed another proof of the functional equation. He borrowed

[redacted]

Property 3: The zeros of $\xi(s)$ have their real parts between 0 and 1.[8]

Proof:

[redacted]

at $s = 1$, the roots of $\xi(s)$ are the same as the roots of $\zeta(s)$. Then as it is proved in Properties 2 that $\zeta(s)$ is zero-free on the half-plane where Re(s) >1, $\xi(s)$ has no root on that half-plane either.

Moreover, the equation $\xi(s) = \xi(1-s)$ implies that $(1-\rho)$ is a root of $\xi(s)$ if and only if ρ is a root of $\xi(s)$. Hence, since it is shown that $\xi(s)$ has no root on the half plane Re(s) >1, $\xi(s)$ does not have any root on the half-plane Re(s) < 0. Therefore, all the roots of $\xi(s)$, if existing, have to lie in the strip $0 \leq \text{Re}(s) \leq 1$. Q. E. D.

Let $s = \sigma + it$ and consider $s > 1$. Then

$$\ln|\zeta(s)| = \mathrm{Re}(\ln \zeta(s)) = \sum_{n=2}^{\infty} c_n n^{-s} \cos(t \ln t),$$

where $c_n = \begin{cases} \frac{1}{m} \text{ if } n = p^m, \ p \text{ is prime} \\ 0 \qquad\qquad \text{otherwise} \end{cases}.$

It follows that

$$\ln\left|\zeta(\sigma)^3 \zeta(\sigma+it)^4 \zeta(\sigma+i2t)\right| = \sum_{n=2}^{\infty} c_n n^{-s}(3 + 4\cos(t \ln n) + \cos(2t \ln n)).$$

Because $3 + 4 \cos t + \cos 2t = 2 + 4 \cos t + 2(\cos t)^2 = 2(1 + \cos t)^2 \geq 0$,

$$\ln\left|\zeta(\sigma)^3 \zeta(\sigma+it)^4 \zeta(\sigma+i2t)\right| \geq 0.$$

Then $\zeta(\sigma)^3 \zeta(\sigma+it)^4 \zeta(\sigma+i2t) \geq 1$.

Thus, $$((\sigma-1)\zeta(\sigma))^3 \left|\frac{\zeta(\sigma+it)^4}{\sigma-1}\right| |\zeta(\sigma+i2t)| \geq \frac{1}{\sigma-1} \qquad (8)$$

for $\sigma > 1$ and for all values of t.

Since $\zeta(s)$ has a simple pole at $s = 1$, we have

$$\lim_{\sigma\to1}(\sigma-1)\,\zeta(s) = 1.$$

Suppose $t \neq 0$ and assume $\zeta(1+it) = 0$.

Then we would have

$$\lim_{\sigma\to1} \frac{\zeta(\sigma+it)}{\sigma-1} = \zeta'(1+it).$$

In addition, $\lim_{\sigma\to1} \frac{1}{\sigma-1} = \infty$. Hence, (8) implies $\lim_{\sigma\to1} |\zeta(\sigma+2it)| = \infty$. And because $\zeta(s)$ has only one simple pole at $s = 1$, it indicates that t has to be 0, which contradicts to the assumption that $t \neq 0$. Therefore, $\zeta(1+it) \neq 0$. Q.E.D.

b. Second step of the proof of the Prime Number Theorem:

- 1: Show that $\Psi(x) \sim x$.[10]

Since no one had been able to prove the theorem directly from $\pi(x)$ till his time, Hadamard decided to approach the theorem indirectly. He used another function that behaves similar to $\pi(x)$ but is easier to estimate in his proof. He introduced the step function $\Psi(x)$, which starts at 0 and has a jump of $\ln p$ at each prime power p^n.[11] So the formula of $\Psi(x)$ is

$$\Psi(x) = \sum_{p^n < x} \ln p \qquad (9)$$

By evaluating the definite integral $\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left[-\frac{\zeta'(s)}{\zeta(s)}\right]\frac{x^s ds}{s}$, Hadamard obtained a representation for $\Psi(x)$:

$$\Psi(x) = x - \sum \frac{x^\rho}{\rho} + \sum \frac{x^{-2n}}{2n} - \frac{\zeta'(0)}{\zeta(0)} \qquad (10)$$

where x > 1 and ρ ranges over the zeros of the Riemann zeta function.

Then $\int^x \Psi(t)dt = \frac{x^2}{}$ [redacted] $- \sum \frac{x^{\rho+1}}{}$ [redacted] $- \sum \frac{x^{-2n+1}}{}$ [redacted] $- \frac{\zeta'(0)}{}$ [redacted] $x + \frac{\zeta'(-1)}{}$ [redacted]. (11)

[redacted]

$\int_0^x \Psi(t)\,dt - x^2/2$ [redacted] $x^{\rho-1}$ [redacted]

[redacted]

* 2: Deduce the Prime Number Theorem[12]

[redacted]

[redacted]

[redacted]

[redacted]

[illegible]

[redacted]

is at most

$$\frac{(1+\varepsilon)y}{\ln y}-\frac{(1-\varepsilon)x}{\ln x}+\int_x^y\frac{(1+\varepsilon)t\,dt}{t(\ln t)^2}=2\varepsilon\frac{x}{\ln x}+(1+\varepsilon)\left\{\left[\frac{t}{\ln t}\right]_x^y+\int_x^y\frac{t\,dt}{t(\ln t)^2}\right\}$$

[redacted]

$$=2\varepsilon\frac{x}{\ln x}+(1+\varepsilon)\left\{\int_x^y\frac{dt}{\ln t}\right\}=2\varepsilon\frac{x}{\ln x}+(1+\varepsilon)[Li(y)-Li(x)]$$

and at least equal to $-2\varepsilon\frac{x}{\ln x}+(1-\varepsilon)[Li(y)-Li(x)]$.

Therefore, for a fixed x, $\frac{\pi(y)}{Li(y)}$ is at most

$$\frac{2\varepsilon\frac{x}{\ln x}+(1+\varepsilon)[Li(y)-Li(x)]+\pi(x)}{Li(y)}=1+\varepsilon+\frac{2\varepsilon\frac{x}{\ln x}+(1+\varepsilon)Li(x)+\pi(x)}{Li(y)}\le 1+2\varepsilon$$

and is at least $1-2\varepsilon$ for sufficiently large y. Because ε is an arbitrary number, this implies $\frac{\pi(y)}{Li(y)}\to 1$, or equivalently, $\pi(y)\sim Li(y)$. The Prime Number Theorem is proved.

Q. E. D.

Other proofs of the Prime Number Theorem:

After the first proof of the Prime Number Theorem by Hadamard and Poussin, more proofs came out; some of them were shorter, but they all involve difficult complex analysis. In 1949, Atle Selberg and Paul Erdős found the first elementary proof. Since the proof avoided the use of complex analysis, it was considered "elementary". However, it was less natural and less intuitive than the proof via Riemann's zeta function while still remaining quite elaborate and more taxing to comprehend. Therefore, the analytic proofs were still preferred.

III Relationship between the Prime Number Theorem and the Riemann Hypothesis:

1. The Riemann Hypothesis:

In his paper on number theory, while studying the zeta function and trying to find a good [illegible]

In order to rewrite this equation under integral form, Riemann defined a new function $J(x)$ that I would like to refer as the Prime-Jumping Function. $J(x)$ is a function that starts at 0

Combining these two, we get:

$$\ln\zeta(s) = \ln\xi(s) - \ln\Pi\left(\frac{s}{2}\right) + \frac{s}{2}\ln\pi - \ln(s-1)$$

$$= \ln\xi(0) + \sum_{\rho}\ln\left(1-\frac{s}{\rho}\right) - \ln\Pi\left(\frac{s}{2}\right) + \frac{s}{2}\ln\pi - \ln(s-1) \qquad (18)$$

[redacted]

[redacted]

[redacted]

[redacted]

Using $\ln \Pi\left(\frac{s}{2}\right)=\sum_{n=1}^{\infty}\left[-\ln\left(1+\frac{s}{2n}\right)+\frac{s}{2}\ln\left(1+\frac{1}{n}\right)\right]$, we can conclude that (19) is equal to

$$\int_{x}^{\infty}\frac{dt}{t(t^2-1)\ln t}.$$ [19]

Forth term: $-\frac{1}{2\pi i}\frac{1}{\ln x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\frac{s}{2}\ln\pi}{s}\right]x^s ds=-\frac{1}{2\pi i}\frac{1}{\ln x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\ln\pi}{2}\right]x^s ds$

$$=-\frac{1}{2\pi i}\frac{1}{\ln x}\int_{a-i\infty}^{a+i\infty}0x^s\,ds=0.$$ [20]

Fifth term: $-\frac{1}{2\pi i}\frac{1}{\ln x}\int^{a+i\infty}\frac{d}{ds}\left[-\ln(s-1)\ldots\right]\ldots\frac{1}{2\pi i}\frac{1}{\ln x}\int^{a+i\infty}\frac{d}{ds}\left[\ln(s-1)\ldots\right]$ [illegible]

[illegible]

- <u>The Expression of $\pi(x)$ in terms of $J(x)$</u>

Based on the definition of $J(x)$, Riemann found a relationship between $\pi(x)$ and $J(x)$

$$J(x) = \pi(x) + \frac{1}{2}\pi(\sqrt{x}) + \frac{1}{3}\pi(\sqrt[3]{x}) + \frac{1}{4}\pi(\sqrt[4]{x}) + \ldots = \sum_{i=1}^{\infty}\frac{1}{i}\pi(\sqrt[i]{x}).$$

Then, by the Mobius Inversion[21], he inverted the order of the equation and got

$$\pi(x) = J(x) - \frac{1}{2}J(\sqrt{x}) - \frac{1}{3}J(\sqrt[3]{x}) - \ldots \qquad (21)$$

The first sum is actually finite for each given x since $x^{1/n} < 2$ for a sufficiently big value of n, which leads to $\pi(x^{1/n}) = 0$. Then it follows that the second series, the representation of $\pi(x)$ in terms of $J(x)$, is finite also.

- <u>The Explicit Formula of $\pi(x)$:</u>

Substituting (20) to (21), Riemann obtained an explicit formula of the Prime-Counting function $\pi(x)$ as he desired. This formula includes 3 types of terms:

- The stable terms, which do not grow as x increases: They are the last two terms in (20).
- [illegible]

b. *Riemann's Approximation of $\pi(x)$*

Because the oscillating terms $\sum_{\operatorname{Im}\rho>0}\left[Li(x^{\rho})+Li(x^{1-\rho})\right]$ are hard to evaluate and some of the terms cancel each other due to opposite signs, Riemann dropped these terms from the formula and suggested an approximation of $\pi(x)$:

[illegible]

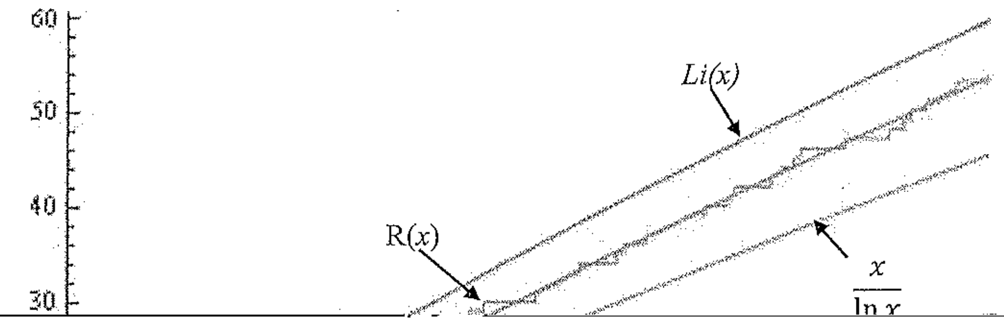
or in a shorter form, $\pi(x) \sim Li(x)+\sum_{n=2}^{\infty}\frac{\mu(n)}{n}Li\left(x^{1/n}\right)$ (22)

where $\mu(n)$ is the Mobius function

$$\mu(n)=\begin{cases}1 \text{ if } n \text{ is a square}-\text{free integer with an even number of prime factors}\\ -1 \text{ if } n \text{ is a square}-\text{free integer with an odd number of prime factors}\\ 0 \quad \text{if } n \text{ is not square}-\text{free}\end{cases}$$

This formula is called the Riemann's formula for $\pi(x)$.

Furthermore, in fact, the first two terms in (22) is the approximation of $\pi(x)$ in the


60
50
40
30
Li(x)
R(x)
x / ln x

c. The error term of the Prime Number Theorem:

In his paper, Riemann also set up the connection between the relative error in the asymptotic approximation of $\pi(x)$ and the distribution of the complex zeros of the Riemann zeta function. Assuming his hypothesis about the nontrivial zeros of the zeta function is true, Riemann was able to give an exact analytical formula for the error of the approximation of $\pi(x)$

$$\pi(x)-\sum_{n=1}^{N}\frac{\mu(n)}{n}Li\left(x^{1/n}\right)=\sum_{n=1}^{N}\sum_{\rho}Li\left(x^{\rho/n}\right)+\text{"some lesser terms"}.$$

Moreover, it is stated that the Riemann Hypothesis is equivalent to a much better error [redacted] strictly inside the strip $0 < \mathrm{Re}(s) < 1$.[22] The Riemann Hypothesis concerns about the non-trivial zeros and asserts that all non-trivial zeros should lie on the same line called the **critical line,** ½ + *it*, where t is a real number and *i* is the imaginary unit.

1. Location of trivial zeros

Since the function $x(e^x - 1)^{-1}$ is analytic near $x = 0$, it can be expanded as a power series

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!} \qquad (23)$$

where the coefficients B_b are called Bernoulli numbers. It is noticed that the odd Bernoulli

[redacted]

[redacted]

been detected. This section will show two techniques of finding the nontrivial roots manually.[24]

a. *Euler-Maclaurin Summation:*

[redacted]

For $\zeta(s)$, applying Euler-Maclaurin summation to the series $\zeta(s) = \sum_1^\infty n^{-s}$, we get[25]

$$\zeta(s) \sim \sum_{n=1}^{N-1} n^{-s} + \frac{N^{1-s}}{s-1} + \frac{1}{2}N^{-s} + \frac{B_2}{2} s N^{-s-1} + \ldots + \frac{B_{2\nu}}{(2\nu)!} s(s+1)\ldots(s+2\nu-2) N^{-s-2\nu+1} \tag{27}$$

where B_i is the Bernoulli polynomial that satisfies $B_n(x+1) - B_n(x) = n x^{n-1}$ (derived from [redacted]

decrease rapidly in the magnitude.

[redacted]

[illegible]

b. The Riemann-Siegel Formula:

[redacted]

approximation of $Z(18)$ as follows

[redacted]

$(t/2\pi)^{1/2} = 1.692569$, so $N = 1$, and $p = 0.692569$.

$$\text{Then } Z(18) \sim 2\cos\vartheta(18) + (-1)^{1-1}\left(\frac{18}{2\pi}\right)^{-1/4} \frac{\cos 2\pi\left(0.692569^2 - 0.692569 - \frac{1}{16}\right)}{\cos(2\pi 0.692569)}$$

$$= 1.993457 + (0.768647)\frac{-0.159022}{-0.353070} = 1.993457 + 0.346197 = 2.339654$$

[redacted]

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