# ON DIRICHLET'S CONJECTURE ON RELATIVE CLASS NUMBER ONE

AMANDA FURNESS

Abstract. We examine relative class numbers, associated to class numbers of quadratic fields $\mathbb{Q}(\sqrt{m})$ for $m > 0$ and square-free. The relative class number is the quotient:

$$H_d(f) = \frac{h(f^2 d)}{h(d)},$$

where $d$ is the discriminant of $\mathbb{Q}(\sqrt{m})$ and $h$ refers to the class number. It is not known if for every $m$ there exists an $f > 1$ for which this ratio is one, although Dirichlet conjectured that this is true. We prove that there does exist such an $f$ when $\sqrt{m}$ has a particular continued fraction form. The main result concerns when the continued fraction is diagonal, i.e. when all entries are equal.

## 1. Introduction

Compared to imaginary quadratic fields when $m < 0$, very little is known about the class number problem for real quadratic fields. Properties of the relative class number for $m > 0$ are even more elusive. An open question in this area is whether there is a relative class number of 1 in every real quadratic field. Dirichlet conjectured that this is true and in this paper we narrow down the possibilities of where it may not hold true by finding a relative class number of 1 for certain values of $m$. The continued fraction expansions of $\sqrt{m}$ follows specific patterns that enable us to guarantee relative class numbers of one for many values of $m$ at once. We use a similar proof for each case although they rapidly become more complicated as the period length of the continued fractions lengthen. We prove Dirichlet's conjecture for continued fraction expansions with period lengths of 1, 2 and 3 as well as all cases where $\sqrt{m} = \langle n; \overline{a, a, \ldots, a, 2n} \rangle$. In addition, we prove the conjecture for some special cases of period lengths 4 and 5.

Sections 2, 3, and 4 will provide the necessary background for Dirichlet's formula for computing the relative class number, which is introduced in Section 5. Section 6 will then give background on continued fractions which will lead in to our research and results in Sections 7 and 8.

## 2. Quadratic Fields

This gives us the equation $39^2 - 5^2 \cdot 61 = -4$, so $(x, y) = (39, 5)$. Thus the fundamental unit for $d = 61$ is $\varepsilon_{61} = \frac{39 + 5\sqrt{61}}{2}$.

## 4. Quadratic Reciprocity

We now introduce the Legendre symbol along with some of its useful properties, which will appear in our main theorem in Section 5.

Definition 4.1. The symbol $\left(\frac{a}{p}\right)$ is called the *Legendre symbol* and is defined as follows:

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p \mid a \\ 1 & \text{if } x^2 \equiv a \pmod{p} \text{ has a solution} \\ -1 & \text{if } x^2 \equiv a \pmod{p} \text{ has no solution} \end{cases}$$

where $a, p \in \mathbb{Z}$ and $p$ prime.

The following theorem presents properties that help us determine the value of the Legendre symbol in difficult cases.

Theorem 4.2. [?] *Let* $a, b, p \in \mathbb{Z}$. *Then the following properties hold:*

(1) *If* $a \equiv b \pmod{p}$ *then* $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$
(2) $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$
(3) *If* $p, q$ *are odd primes with* $p \neq q$*, then*

$$\Big( q$$

where $h\hat{}f^{2}d\bullet$ refers to the class number de nition above but with ; > $0_{f}$. It should be clear that $h\hat{}d\bullet$ B $h\hat{}f^{2}d\bullet$ since $0_{f}$ b 0. Rather than computing this ratio directly, we have a formula to compute the relative class number shown in the following theorem from Dirichlet.

Theorem 5.1 (Dirichlet). [?] *Let* m *be a xed, square-free, positive integer, and* d *be the eld discriminant of* $Q\hat{}\ \overline{m}\bullet$. *De ne* $\hat{}f\bullet$ $f \prod_{q\S f} \langle 1 - \langle \frac{d}{q}\bullet \frac{1}{q}\bullet$ *where* $\langle \frac{d}{q}\bullet$ *is the Legendre symbol and* q *is prime. De ne* $\hat{}f\bullet$ *to be the smallest positive integer such that* $\hat{}"_{m}\bullet^{\phi\hat{}f\bullet} > 0_{f}$, *i.e.* $\hat{}"_{m}\bullet^{\phi\hat{}f\bullet}$ $\frac{x+y\ \overline{m}}{z}$ *where* y 0 ˆ*mod* f •. *Then*

$$H_{d}\hat{}f\bullet \quad \frac{\hat{}f\bullet}{\hat{}f\bullet}$$

Dirichlet conjectured that for every m and corresponding d there exists an f such that $H_{d}\hat{}f\bullet$ 1, although this remains an open question. We examine this problem by looking at the continued fraction expansions of o $\overline{m}$ for certain values of m, which will be preceded by some background on continued fractions.

## 6. Continued Fractions

We begin by de ning the type of continued fraction we are interested in.

De nition 6.1. The *in nite periodic continued fraction* denoted $a a_{0}; \ldots; a_{n}; \overline{b_{0}; \ldots; b_{m}} f$ is equal to
$`a_{0}; \ldots; a_{n}; b_{0}; \ldots; b_{m}; b_{0}; \ldots; b_{m}; b_{0}; \ldots e$

$$a_{0} + \cfrac{1}{\ldots + \cfrac{1}{a_{n} + \cfrac{1}{b_{0} + \cfrac{1}{\ldots + \cfrac{1}{b_{m} + \cfrac{1}{b_{0} + \ldots}}}}}}$$

More speci cally, o $\overline{m}$ has a certain form of in nite periodic continued fraction expansion.

Theorem 6.2. [?] *The continued fraction expansion of* o $\overline{m}$ *for a positive integer* m *that is not a perfect square is* $a n; \overline{a_{1}; a_{2}; \ldots; a_{2}; a_{1}; 2n} f$ *where* n o $\overline{m}$.

De nition 6.3. [?] For any continued fraction $`a_{0}; a_{1}; a_{2}; \ldots e$, a *convergent* $\frac{h_{i}}{k_{i}} \quad \frac{a_{i}h_{i-1} + h_{i-2}}{a_{i}k_{i-1} + k_{i-2}}$ where i C0 and $h_{-2}$ 0, $h_{-1}$ 1, $k_{-2}$ 1, and $k_{-1}$ 0.

## 7. Main Results

Our main result is the proof of a relative class number of one for all $m$ values such that $\sqrt{m} = [n; \overline{a, a, \ldots, a, 2n}]$. We lead up to this theorem with three simple cases $[n; \overline{2n}]$, $[n; \overline{a, 2n}]$, and $[n; \overline{a, a, 2n}]$. We approach each of these simple cases by finding a general form of $m$, using our algorithm to find the fundamental unit, then proving the existence of an $f$ that gives a relative class number of one.

### 7.1. The Base Case $[n; \overline{2n}]$.

We will examine the continued fraction expansions of $\sqrt{m}$, beginning with the simplest case $[n; \overline{2n}]$. We start by solving the continued fraction for a general form of $m$.

**Lemma 7.1.** *The continued fraction* $[n; \overline{2n}] = n + \frac{1}{2n + \frac{1}{2n+\ldots}}$, *is equal*

| $i$ | -2 | -1 | 0 | 1 |
|---|---|---|---|---|
| $a_i$ | | | $n$ | $2n$ |
| $h_i$ | 0 | 1 | $n$ | |
| $k_i$ | 1 | 0 | 1 | |

This gives us the equation $n^2 - 1^2 \cdot m = -1$. So if $m \equiv 2, 3 \pmod 4$, the fundamental unit for $m = n^2 + 1$ is $\varepsilon = n + \sqrt{m}$. If $m \equiv 1 \pmod 4$, we get $(2n)^2 - (2 \cdot 1)^2 \cdot m = -4$ so the fundamental unit is $\varepsilon = \frac{2n + 2\sqrt{m}}{}$

$$2nx + 1 = 2an + ax^{-1} + 1$$
$$2n = 2anx^{-1} + ax^{-2}$$
$$ax^{-2} + 2anx^{-1} - 2n = 0$$
$$x^{-1} = \frac{-2an \pm \sqrt{4a^2n^2 + 4(2an)}}{2a}$$
$$= -n \pm \sqrt{n^2 + \frac{2n}{a}}$$

Disregarding the negative solution gives us $[n; \overline{a, 2n}] = n + x^{-1} = n + \left(-n + \sqrt{n^2 + \frac{2n}{a}}\right) = \sqrt{n^2 + \frac{2n}{a}}$. Since $m \in \mathbb{Z}$ and $m$ is square-free, we must have that $a \mid 2n$ and $n^2 + \frac{2n}{a}$ is not a perfect square. □

Next we find the general form of the fundamental unit.

Lemma 7.5.

$a^2+2)$



*Proof.* Let $x = [\overline{a;a;2n}]$. So $[n;\overline{a;a;2n}] = n + x^{-1}$. Note that $x = [\overline{a;a;2n}] = [a;a;2n;x]$. So

$$x = a + \cfrac{1}{a + \frac{1}{2n+x^{-1}}}$$

$$x = a + \cfrac{1}{\frac{2an+ax^{-1}+1}{2n+x^{-1}}}$$

$$x = a + \frac{2n+x^{-1}}{2an+ax^{-1}+1}$$

$$2anx + a + x = 2a^2n + a^2x^{-1} + a + 2n + x^{-1}$$

$$(a^2+1)x^{-2} + (2a^2n+2n)x^{-1} - (2an+1) = 0$$

$$x^{-1} = \frac{-2a^2n - 2n \pm \sqrt{(2a^2n+2n)^2 + 4(a^2+1)(2an+1)}}{2(a^2+1)}$$

$$= -n \pm \frac{1}{2(a^2+1)}\sqrt{n^2(2a^2+2)^2 + 2(2a^2+2)(2an+1)}$$

$$= -n \pm \sqrt{n^2 + \frac{2an+1}{a^2+1}}$$

Disregarding the negative solution gives us $[n;\overline{a;a;2n}] = n + x^{-1} = n + \left(-n + \sqrt{n^2 + \frac{2an+1}{a^2+1}}\right)$

This gives us the equation $(a^2n + a + n)^2 - (a^2 + 1)^2 m = -1$ so the fundamental unit for $m = n^2 + \frac{2an+1}{a^2+1}$ is $\varepsilon_m = (a^2n+a+n)+(a^2+1)\sqrt{m}$. □

We now provide the criteria for $a, n$ that will give us a relative class number of 1, which follows the pattern introduced in the first two

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So

$$\frac{P_r}{Q_r} \qquad \frac{k_r}{h_{r-1}} \qquad \frac{2n\, k_{r-1} + k_{r-2}}{h_{r-1}}$$

$$\frac{2n\hat{\ } ak_{r-2} + k_{r-3}\bullet + ak_{r-3} + k_{r-4}}{ah_{r-2} + h_{r-3}}$$

$$\frac{a\hat{\ } 2n\, k_{r-2} + k_{r-3}\bullet + 2n\, k_{r-3} + k_{r-r}}{ah_{r-2} + h_{r-3}}$$

$$\frac{a\, P_{r-1} + P_{r-2}}{a\, Q_{r-1} + Q_{r-2}}$$

Thus $P_r$

*Proof.* We know that since $\frac{P_r}{Q_r} \in \mathbb{Z}$, $P_r \equiv 0 \pmod{Q_r}$. By reducing the formula in the previous theorem modulo $Q_r$ we get

$$0 \equiv 2nQ_{r-1} - aQ_{r-1} \pmod{Q_r}.$$

So $2nQ_{r-1} \equiv aQ_{r-1} \pmod{Q_r}$ and thus $2n \equiv a \pmod{Q_r}$. □

With this relationship between $n$ and $a$, we can now find a general form of $n$ that will guarantee the existence of a prime that will give us a relative class number of one.

**Lemma 7.15.** *Let* $m = [\overline{an; a; a; \dots; a; 2n}] = \sqrt{n^2 + \frac{P_r}{Q_r}}$. *Then there exists a prime* $f$ *such that* $f \mid m$ *and* $f \nmid Q_r$.

*Proof.* By the previous lemma, $2n \equiv a \pmod{Q_r}$. So we have three possibilities: $a$ and $Q_r$ are both even, $a$ and $Q_r$ are both odd, or $a$ is even and $Q_r$ is odd. These give us the following values of $n$:

$$n = \begin{cases} \frac{a}{2} + l \cdot \frac{Q_r}{2}, \text{ for some } l \geq 0 & \text{when } a \text{ and } Q_r \text{ are both even} \\ \frac{a + Q_r}{2} + l \cdot Q_r, \text{ for some } l > 0 & \text{when } a \text{ and } Q_r \text{ are both odd} \\ \frac{a}{2} + l \cdot Q_r, \text{ for some } l \geq 0 & \text{when } a \text{ is even and } Q_r \text{ is odd} \end{cases}$$

In each of these cases, we get $m = n^2 + \frac{P_r}{Q_r} \geq Q_r$. Thus there must exist a prime that divides m25 11.9552 Tf 1418-19.TJ/846 7.9701 Tf 9.272 -1.799 Td [(r)]TJ/

complete all these steps for the general case and will therefore look at only a few special cases.

**Theorem 8.1.** *The continued fraction* $[n; \overline{a; b; a; 2n}]$ *is equal to*

$$\sqrt{n^2 + \frac{2abn+2n+b}{a^2b+2a}}$$

*where* $a^2b+2a \mid 2abn+2n+b$ *and* $n^2+\dfrac{2abn+2n+b}{a^2b+2a}$ *is not a perfect square.*

The proof of this theorem is of the same form as the corresponding proofs in the previous cases but is omitted here due to the complexity of the resulting quadratic equation.

We continue with this case by finding the fundamental unit.

**Theorem 8.2.** *Let* $\sqrt{m} = [n; \overline{a; b; a; 2n}] = \sqrt{n^2 + \frac{2abn+2n+b}{a^2b+2a}}$. *Then the fundamental unit* $\varepsilon_m = (a^2bn + a^2 + b^2 + an + 1) + (a^2b + 2a)\sqrt{m}$.

*Proof.* □

We simplify this into a few special subcases, looking at only particular values of $a$ and $b$.

In each case, we have simplified the problem to proving that there exists a prime $f$ that divides $m$ and does not divide the denominator of the fraction of the general form of $m$

Now, sincek >Z,

$$2an + 2n + 1 \quad 0 \hat{} \text{mod } a^2 + 2a\bullet$$

$$2n\hat{}a + 1\bullet \quad -1 \hat{} \text{mod } a^2 + 2a\bullet$$

$$-a^2 - 2a - 1 \hat{} \text{mod } a^2 + 2a\bullet$$

$$-\hat{}a + 1\bullet^2 \hat{} \text{mod } a^2 + 2a\bullet$$

$$n \quad -\text{‹} \; a + 1$$

**Corollary 8.9** (of Theorem ??). *If* $\sqrt{m} = [n; \overline{2, b, 2, 2n}] = \sqrt{n^2 + \frac{4bn+2n+b}{4b+4}}$*, then the fundamental unit* $\varepsilon_m = (4bn+2n+b^2+5) + (4b+4)\sqrt{m}$.

We now guarantee a relative class number of 1 in the same manner as the previous three subcases.

**Theorem 8.10.** *If* $\sqrt{m} = [n; \overline{2, b, 2, 2n}] = \sqrt{n^2 + \frac{4bn+2n+b}{4b+4}}$*, then* $b$ *is even and there exists a prime* $p$ *such that* $p \mid m$ *and* $p \nmid 4b+4$.

*Proof.* Let $k = (4b+4)/4$

Additionally, we will immediately prove the existence of an $f$ that divides $m$ and does not divide the denominator of the corresponding fraction in the general form of $m$, with the assumption that this denominator is the $y$-term of the fundamental unit and will therefore give us a relative class number of 1 by the logic provided in previous cases.

Theorem 8.12. *If* $^{o}$ —

Future research in this area might include continuing our proof technique for larger continued fraction expansions, finding a way to generalize the $[a_n; \overline{a; b; a; 2a_n}]$ and $[a_n; \overline{a; b; b; a; 2a_n}]$ cases based on the simple subcases we solved (i.e. by induction), or finding a new way to handle these larger cases.

## References

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